ON THE FLOW IN A DIFFUSOR IN THE PRESENCE OF A MAGNETIC FIELD

(O TECHENII V DIFFUZORE V PRISUTSTVEE MAGNITNOGO POLIA)

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1. The steady flow of a viscous incompressible fluid in the presence of a magnetic field $\vec{H}$ is described by a system of magneto-hydrodynamic equations

$$(\mathbf{v}\Delta)\,\mathbf{v} = \nabla\left(\frac{p_0}{\rho} - \frac{p}{\rho}\right) + \nu\Delta\mathbf{v} + \operatorname{rot}\mathbf{h}\times\mathbf{h}$$

$$\operatorname{div}\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{h} = 0, \qquad \operatorname{rot}(\mathbf{v}\times\mathbf{h}) + \nu_m\Delta\mathbf{h} = 0 \qquad (1.1)$$

$$\left(\mathbf{H} = \frac{\mathbf{h}}{\sqrt{4\pi\rho}},\ \nu_m = \frac{c^2}{4\pi\sigma}\right)$$

Here ν_m is the "magnetic" viscosity, σ is the conductivity of the medium, c is the velocity of light in vacuum, p_0 is the pressure at any given fixed point of the flow; the remaining symbols are conventional.

The system (1.1) serves for determining the velocity $\vec{v}$, the quantity $\vec{h}$, having the dimension of velocity, and the quantity $(p_0 - p)/\rho$.

In the case of motion in a plane, the independent variables would be the distance from the origin of coordinates r, and the polar angle θ, and the basic parameters would be the coefficients of kinematic and magnetic viscosity ν and ν_m.

We assume that the flows considered are completely determined by the indicated set of parameters, several dimensionless constants ξ, and also $Q_1, \ldots, Q_n$ having dimensions

$$[Q_i] = L^{p_i}T^{q_i}$$

Then, by use of the relations of the theory of similarity and dimensions [1], the quantities to be determined v_r, v_θ, h_r, h_θ; and $(p_0-p)/\rho$

may be expressed as

$$v_r = \frac{\nu}{r} f, \quad v_\theta = \frac{\nu}{r} \varphi, \quad h_r = \frac{\nu}{r} \psi, \quad h_\theta = \frac{\nu}{r} \beta, \quad \frac{p_0 - p}{\rho} = \frac{\nu^2}{r^2} F \tag{1.2}$$

Here p_0 is the pressure at infinity; the functions f, ϕ, ψ, β, and F depend on the dimensionless quantities θ, ν_m/ν, ξ, δ_1, ..., δ_n

$$\delta_i = Q_i r^{-(p_i + 2q_i)} \nu^{q_i}$$

Just as in the book [1], we assume that $p_i + 2q_i = 0$. Then the dimensions of the quantities Q_i consist of some power of the dimensions of the coefficient of kinematic viscosity ν, and the functions f, ϕ, ψ, β and F do not depend on r. Setting the Equations (1.2) into the System (1.1), we obtain the following system of ordinary differential equations:

$$\varphi = \varphi_0 \equiv \text{const}, \qquad \beta = \beta_0 \equiv \text{const}, \qquad f\beta_0 - \psi\varphi_0 + \nu_m\psi' / \nu = 0$$

$$f'' - \varphi_0 f' + f^2 + \varphi_0^2 - 2F + \beta_0\psi' = 0, \qquad F' + 2f' - \psi\psi' = 0 \tag{1.3}$$

Integration of the last equation of the System (1.3) gives

$$F + 2f - \tfrac{1}{2}\psi^2 = C \qquad (C = \text{const}) \tag{1.4}$$

Eliminating F, we obtain a system of the third order for determination of the functions $f(\theta)$ and $\psi(\theta)$

$$f'' - \varphi_0 f' + f^2 + 4f + \beta_0\psi' - \psi^2 + \varphi_0^2 - 2C = 0$$

$$f\beta_0 - \psi\varphi_0 + \nu_m\psi' / \nu = 0 \tag{1.5}$$

Eliminating $f(\theta)$, we can reduce the System (1.5) to one nonlinear equation of the third order with respect to the function $\psi(\theta)$.

Using the generalized Ohm's law, it is possible to show that the type of flow considered has an electric field vector equal to zero. It is possible to treat the motion as the flow of a liquid in a magnetic field with a source at the origin of coordinates flowing perpendicularly to the plane of r, θ. In this case, according to the law of Biot-Savart, $h_\theta = h_0/r$ and the constant h_0 has the dimension of the coefficient of kinematic viscosity.

2. We consider the flow from a source (or sink) between two plane walls tilted with respect to each other at an angle α (diffusor). We assume an infinitely conducting medium. The magnetic field induces a current I_0, flowing along the vertex of the diffusor angle. If the circulation velocity v_θ vanishes, the propagation of the liquid into the region occupied by the field is impossible, as follows from the second equation of (1.5): since $\nu_m = 0$, $\beta_0 \neq 0$, $\phi_0 = 0$, we obtain $f' \equiv 0$. If we assume the possibility of the circulation velocity having the value $v_\theta = \Gamma / 2\pi r$ (for example, in the walls of the angle holes are made so as to permit liquid motion normal to the walls), then flow from the source (or out of the sink) can take place. For the function $f(\theta)$ we obtain from the System (1.5) an equation of the second order

$$f'' + f'(\beta_0^2 - \varphi_0^2)/\varphi_0 - f^2(\beta_0^2 - \varphi_0^2)/\varphi_0^2 + 4f + \varphi_0^2 - 2C = 0 \qquad (2.1)$$

To solve this equation and determine the constant C, the condition is used that the liquid sticks to the wall, and also the discharge Q through the diffusor is specified

$$f(\pm {}^1/_2\alpha) = 0, \qquad \int_{-\alpha/2}^{+\alpha/2} f\, d\theta = \frac{Q}{\nu} \qquad (2.2)$$

The solution of the problem is given by the formulas

$$v_r = \frac{\nu}{r} f(\theta), \qquad v_\theta = \frac{\Gamma}{2\pi r}, \qquad h_r = \frac{4\pi\nu I_0}{c\Gamma\sqrt{4\pi\rho r}} f(\theta), \qquad h_\theta = \frac{2I_0}{c\sqrt{4\pi\rho r}}$$
$$\frac{p_0 - p}{\rho} = \frac{\nu^2}{r^2}\left(C - 2f + \frac{2\pi I_0^2 f^2}{c^2\rho\Gamma^2}\right) \qquad (2.3)$$

The quantities β_0 and ϕ_0 which enter Equation (2.1) are expressed through the current I_0 and circulation Γ

$$\varphi_0 = \Gamma / 2\pi\nu, \qquad \beta_0 = 2I_0 / c\nu\sqrt{4\pi\rho}$$

Equation (2.1) becomes especially simple in the case (d),

$$\beta_0 = \varphi_0 \quad (\Gamma = \sqrt{4\pi} I_0 / c\sqrt{\rho})$$

and is easily integrated. For the velocity and the family of stream lines the expressions are obtained

$$v_r = \frac{Q}{r}\,\frac{\cos 2\theta - \cos\alpha}{\sin\alpha - \alpha\cos\alpha}, \qquad r(\theta) = A\exp\left(\frac{\pi Q}{\Gamma}\,\frac{\sin 2\theta - 2\theta\cos\alpha}{\sin\alpha - \alpha\cos\alpha}\right)$$

As is evident from (2.3), the vectors of velocity and magnetic field strength are collinear.

3. The flow of a fluid with finite electrical conductivity into a diffusor with opening angle α is described by the system (1.5) in which it is necessary to set $\phi_0 = 0$ (there is only radial velocity). For the solution of this system of the third order, and determination of the constant C which enters into Expression (1.4) for the determination of pressure, four conditions are necessary. Three of them are adherence of

the flow to the walls and the specification of the discharge rate (2.2). For the fourth condition, imposed on the induced magnetic field ψ, it is possible, for example, to assume that the component H_r of the magnetic field is equal in magnitude and opposite in sign on the line $r=$ constant, at the points where the line intersects the diffusor.

We introduce the designations

$$\frac{|Q|}{\nu}=R, \qquad \frac{|Q|}{\nu_m}=R_m, \qquad \frac{\beta_0{}^2\nu}{4\nu_m}=M^2$$

Here R has the meaning of the hydrodynamic Reynolds number, r_m is the magnetic Reynolds number, M is the Hartmann number. In view of the fact that for the quantities $f(\theta)$ and $\psi(\theta)$ which enter the System (1.5), it is proper to set $f(\theta)\sim R$, $\psi(\theta)\sim\beta_0R_m$ it is convenient to go over to the functions $u(\theta)$ and $\lambda(\theta)$ according to the formulas

$$f(\theta)=Ru(\theta), \qquad \psi(\theta)=R_m\beta_0\lambda(\theta)$$

Equations (1.5) and the boundary conditions are then reduced to the form

$$u''+Ru^2+4u+4M^2\lambda'-4R_mM^2\lambda^2-D=0, \qquad u+\lambda'=0$$
$$u\left(\pm\tfrac{1}{2}\alpha\right)=0, \qquad \lambda\left(+\tfrac{1}{2}\alpha\right)=-\lambda\left(-\tfrac{1}{2}\alpha\right), \qquad \int_{-1/2\alpha}^{1/2\alpha}u(\theta)\,d\theta=\pm 1 \qquad (3.1)$$

In the last expression of (3.1) the upper sign corresponds to a source, and the lower one, to a sink. The velocity, magnetic field, and pressure are computed by use of the formulas

$$v_r=\frac{|Q|}{r}u(\theta), \qquad H_0=\frac{\sqrt{4\pi\rho}\nu\beta_0}{r}, \qquad \beta_0=\frac{2I_0}{c\nu\sqrt{4\pi\rho}}$$
$$H_r=\frac{R_m\beta_0\nu\sqrt{4\pi\rho}}{r}\lambda(\theta), \qquad \frac{p_0-p}{\rho}=\frac{\nu^2R}{r^2}\left(\frac{D}{2}-2u+2M^2R_m\lambda^2\right) \qquad (3.2)$$

Finally the case will be considered of small conductivity $R_m\ll 1$, where in the first equation of (3.1) and the relations (3.2) it is possible to neglect terms with the coefficient R_mM^2. Now the magnitude M^2 in the case of strong external fields can be large. The motion is described by the system

$$u''+Ru^2+4(1-M^2)u-D=0, \quad u\left(\pm\tfrac{1}{2}\alpha\right)=0, \quad \int_{-1/2\alpha}^{1/2\alpha}u\,d\theta=\pm 1 \qquad (3.3)$$

and the pressure is expressed by

$$\frac{p_0-p}{\rho}=\frac{\nu^2R}{r^2}\left(\frac{D}{2}-2u\right) \qquad (3.4)$$

Equation (3.3) is integrated in terms of elliptical functions. We are restricted to the simplest cases.

4. We consider the flow of a fluid where the Reynolds number R is small. Equation (3.3) is simplified and transformed into the Equation

$$u'' + 4(1 - M^2)u = D \tag{4.1}$$

For $M^2 < 1$, we obtain for the function $u(\theta)$ and the constant D the expressions

$$u = \pm \omega \frac{\cos 2\omega\theta - \cos \omega\alpha}{\sin \omega\alpha - \omega\alpha \cos \omega\alpha}, \qquad D = \mp \omega \frac{4\omega^2 \cos \omega\alpha}{\sin \omega\alpha - \omega\alpha \cos \omega\alpha} \tag{4.2}$$

$$(\omega^2 = 1 - M^2, \ \alpha \leqslant \pi / \omega)$$

For $M^2 = 1$, the solution of Equation (4.1) has the form

$$u = \pm \frac{3}{2\alpha^3}(\alpha^2 - 4\theta^2), \qquad D = \mp \frac{12}{\alpha^3} \tag{4.3}$$

For $M^2 > 1$, the solution of Equation (4.1) is given by the formulas

$$u = \pm \omega \frac{\operatorname{ch} \omega\alpha - \operatorname{ch} 2\omega\theta}{\alpha\omega \operatorname{ch} \omega\alpha - \operatorname{sh} \omega\alpha}$$

$$D = \mp \omega \frac{4\omega^2 \operatorname{ch} \omega\alpha}{\alpha\omega \operatorname{ch} \omega\alpha - \operatorname{sh} \omega\alpha} \tag{4.4}$$

$$(\omega^2 = M^2 - 1)$$

The function $u(\theta)$ for different Hartmann numbers, for the angle $\alpha = \pi/2$ is shown in Fig. 1. The magnetic field, exerting a drag on the flowing liquid, produces a flatter velocity profile. In the limiting case of strong magnetic field forces ($M^2 \gg 1$) from (4.4) and (3.4), we obtain for the core of the flow approximately

$$v_r = Q / \alpha r, \qquad (p - p_0) / \rho = \pm 2Q^2M^2 / \alpha R r^2 \tag{4.5}$$

We notice that the expressions (4.5) are kernels of the solution of the system

$$\frac{dp}{dr} + \frac{\sigma H_\theta^2}{c^2\rho} v_r = 0, \qquad \frac{d}{dr} r v_r = 0, \qquad H_\theta = \frac{2I_0}{cr}$$

Actually, by virtue of the inequality $R \ll 1$ the inertial forces may be considered negligible compared with the viscous forces; and by virtue of $M^2 \gg 1$, the viscous forces in the kernel of the flow are less than the force of magnetic drag.

Now the induced magnetic field may not be considered in view of the assumption $R_m \ll 1$. In the immediate proximity of the wall, where the forces of friction are considerable, the speed decreases toward zero in conformity with (4.4). Diverging flow in the case considered is characterized by a large pressure gradient that is negative; converging flow is characterized by a large positive pressure gradient.

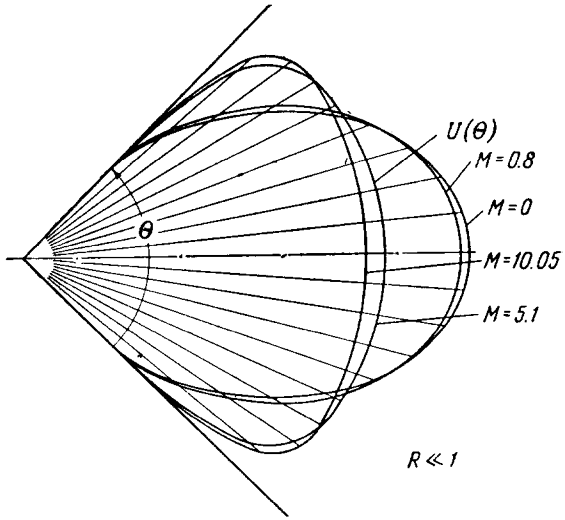

Fig. 1.

5. From the theory of the motion of a nonconducting gas in a diffusor it is known that a symmetrical diverging flow is possible only for Reynolds numbers less than some critical value $R = R_*$, for which friction on the walls of the diffusor is reduced to zero. Interaction of a conducting fluid with the magnetic field leads to an increase in the critical Reynolds number. For determination of the number R_* from Equation (3.3), omitting a cumbersome calculation, it is possible to obtain

$$R_*\alpha = 24F(k, \pi/2)\,[E(k, {}^1\!/_2\,\pi) - F(k, {}^1\!/_2\,\pi)(1-k^2)] \tag{5.1}$$

The parameter k is found from the equations

$$\begin{aligned} &\alpha\sqrt{1-M^2} = 2\sqrt{1-2k^2}F(k, {}^1\!/_2\,\pi) && \text{for } M^2 < 1 \\ &\alpha\sqrt{M^2-1} = 2\sqrt{2k^2-1}F(k, {}^1\!/_2\,\pi) && \text{for } M^2 > 1 \\ &k^2 = 0.5 && \text{for } M^2 = 1 \end{aligned} \tag{5.2}$$

$$F(k, \varphi) = \int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad E(k, \varphi) = \int_0^\varphi \sqrt{1-k^2\sin^2\varphi}\,d\varphi$$

The dependence of R_* on the Hartmann number M^2 for three values of the divergence angle α, obtained from (5.1), is shown in Fig. 2. The velocity profile for $\alpha = \pi/4$, $M^2 = 14.6$ ($R_* = 43.3$) is shown in Fig. 3. The critical value of Reynolds number increases with increase of the Hartmann number, and is reduced with increase in the angle.

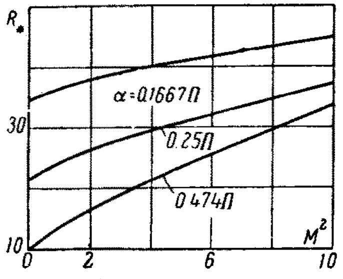

Fig. 2.

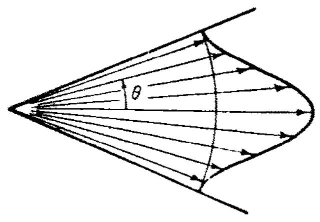

Fig. 3.

6. Symmetrical diverging flow, which does not occur at large Reynolds numbers in the absence of a magnetic field, is possible for large Reynolds numbers and Hartman numbers: $R \gg 1$, $M^2 \gg 1$, $M^2 \sim R$.

Investigating Equation (3.3) for the given assumptions, and omitting a long computation, it is possible to obtain an expression for the velocity profile and the pressure

$$v_r = \frac{Q}{r\alpha}\left[1 - \frac{4(\gamma-3)}{(\sqrt{\gamma-2}+\sqrt{\gamma-3})^2 e^{2\psi} + (\sqrt{\gamma-2}-\sqrt{\gamma-3})^2 e^{-2\psi} - 2}\right] \qquad (6.1)$$

$$\gamma = \frac{6\alpha(M^2-1)}{R} = O(1), \qquad \psi = (\alpha/2-\theta)\sqrt{\frac{R(\gamma-3)}{6\alpha}}, \qquad \frac{p-p_0}{\rho} = \frac{(2\gamma-3)Q^2}{6\alpha^2 r^2}$$

The indicated flow is possible if $6\alpha(M^2-1)/R > 3$. From the expression (6.1) it is evident that the flow has a negative pressure gradient, and the velocity of almost the entire current is equal to the value $Q/r\alpha$, falling toward zero only at the wall. The result for the core of the flow*

$$v_r = \frac{Q}{\alpha r}, \qquad \frac{p-p_0}{\rho} = \frac{(2\gamma-3)Q^2}{6\alpha^2 r^2}$$

* For large Reynolds numbers the flow in the diffuser, in the framework of the theory of the boundary layer was investigated by Gaylitis [3].

is a solution of the system

$$v_r \frac{dv_r}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{\sigma H_\theta^2}{c^2\rho} v_r, \qquad \frac{dv_r r}{dr} = 0, \quad H_\theta = \frac{2I_0}{cr}$$

since the viscous forces in the core of the flow may be neglected as a consequence of $M^2 \gg 1$, and in view of the relation $M^2 \sim R$ the influence of the magnetic field and the inertial terms have the same order of magnitude. The expression for the velocity v_r in (6.1) gives the distribution of velocity in the boundary layer formed. Figure 4 shows the velocity for $R = 400$, $\gamma = 4$, for an angle of $\pi/2$.

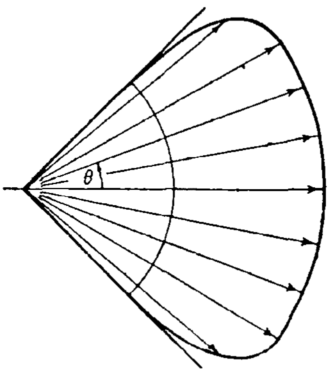

Fig. 4.

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